Kinematic Characterisation of a 3-PUU Parallel Robot

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Abstract
The kinematics of a parallel translating mechanism of the 3-PUU kind is completely developed, with the closed form solution of both position and velocity problems. The possible occurrence of singular configurations is discussed and some hints are provided for the design of singularity free machines. The mechanism is characterised by a radial setting of frame slideways, which leads to kinematics properties different from the corresponding mechanisms that have been studied in literature so far.

1 Introduction
The paper discusses the main kinematics properties of a translational parallel machine (TPM) whose mechanical structure is based on the 3-PUU mechanism: such an architecture is composed by a mobile platform connected to the base by three identical legs with PUU topology, i.e. an actuated prismatic pair is followed by two universal joints. The 3-PUU mechanism has been studied already in past years, even if the 3-UPU variation has been treated more thoroughly [1] and proposed in many different forms [2-3] due to its interesting kinematic properties. On the other hand, the 3-RRU architecture [4] is conceptually similar but has a different and complex kinematics.

The 3-PUU mechanism was first listed by Tsai in [5], where he enumerated all the 3-dof mechanisms composed by prismatic, revolute and spherical pairs; again Tsai and Joshi in 2002 outlined the main kinematic relations of a 3-PUU machine [6] but Giberti et al. had actually built a prototype pneumatic machine based on such a concept already in 2001 [7]. Nevertheless, not all the possible TPMs with such an architecture have been fully characterised, let alone the 3-PUU index j increases from the base to the platform; the limb is connected to the slider at Ai, while Bi and Ci are located at the intersection of the axes of the two revolute pairs of the lower and upper universal joints respectively. Finally, the (fixed) distances among the 3 points Ai, Bi, and Ci are called hi and li respectively.

Some reference frames are needed for the following analysis: the global frame (O,x,y,z) is located at the centre of the fixed base, with z axis normal to the platform, x axis aligned with the direction of the first slideway and y axis set accordingly. Moreover, a local frame (P,u,v,w) is defined at the centre P of the mobile platform, with u directed towards Ci, w normal to the platform itself and v directed accordingly.

of the joints, the mechanism can present translational singularities inside the workspace or even become liable to rotations, as shown for instance by Di Gregorio and Parenti-Castelli [2] and by Bonev and Zlatanov [10], so a careful kinematic analysis must be performed on the concepts beforehand and optimisation is usually required for machine design.

The 3-PUU mechanism discussed in the present paper is characterised by a radial disposition of the frame slideways, that is the three actuated prismatic pairs that connect the legs with the fixed base all lie on the same plane and stem from a common centre, thus presenting a star setting [11]: this is the first time such a mechanism is studied and the resulting kinematic properties (i.e. workspace, singularities, etc.) are quite different from all other configurations that have been studied before, that are characterised by a vertical setting of the slideways, even if in various different dispositions.

2 Mechanical architecture
Figure 1 shows a sketch of the 3-PUU mechanism that has been studied and a detailed view of the i-th leg: wi,j = 1,2,3,4, i = 1,2,3 is the unit vector of the axis of the j-th revolute pair of the i-th leg with the convention that the index j increases from the base to the platform; the limb is connected to the slider at Ai, while Bi and Ci are located at the intersection of the axes of the two revolute pairs of the lower and upper universal joints respectively. Finally, the (fixed) distances among the 3 points Ai, Bi, and Ci are called hi and li respectively.

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In the paper, only the symmetric configuration of Fig. 1a is studied: therefore, since the three slideways are set on the base plane at an equal offset of 120° one from the other, the angle $\phi_i$ from the $x$ to the $x_i$ axis is $\phi_i = 120° \cdot (i-1)$ and the common distance between points $C_i$ and the centre $P$ is $r$; moreover, all the legs have the same dimensions: $h_1 = h_2 = h_3 = h$ and $l_1 = l_2 = l_3 = l$. The platform is initially assembled in such a way that the tool frame is parallel to the global frame: this is also the “zero” configuration for the measurement of the 5 articular coordinates of the legs: $s_i$, $\theta_{1i}$, $\theta_{2i}$, $\theta_{3i}$, $\theta_{4i}$, $i=1,2,3$; it is noted that the only actuated variables are $s_i$.

Such 3-PUU mechanism becomes a TPM provided that the following two conditions are satisfied for each leg ($i=1,2,3$):

$$ii32ww = 0$$

$$ii41ww = (1-2)$$

In fact, the angular velocity of the platform, $\omega$, can be written in the following three different ways for $i=1,2,3$:

$$\omega = \sum_{j=1}^{4} \dot{\theta}_j w_j$$

(3)

that, considering (1) and (2) can be re-written as:

$$\omega = (\dot{\theta}_i + \dot{\theta}_4) w_{2i} + (\dot{\theta}_{2i} + \dot{\theta}_{3i}) w_{2j}$$

(4)

The algebraic system (4) of 9 scalar equations in the 9 unknowns: $a_0$, $a_0$, $a_0$, $(\dot{\theta}_i + \dot{\theta}_4)$, $(\dot{\theta}_{2i} + \dot{\theta}_{3i})$, $i=1,2,3$ only admits the trivial null solution, provided that system’s configuration is not singular (see §4.1 further on). Since the same considerations can be worked out for platform’s angular acceleration $\ddot{\omega}$, it can be concluded that the platform, out of singular configurations, can only translate.

3 Position kinematics

One loop-closure equation can be written for each leg $i=1,2,3$, as follows, Fig. 2:

$$\overrightarrow{OA_i} + \overrightarrow{A_iB_i} + \overrightarrow{B_iC_i} = \overrightarrow{OP} + \overrightarrow{PC_i}$$

(5)

Figure 2. Loop-closure of one leg

Only the last term at the left hand side of Eq. (5) deserves a little more attention:

$$\overrightarrow{B_iC_i} = l\cos\theta_{2i}\left(\overrightarrow{w_{2i}} \times \overrightarrow{w_{4i}}\right) - l\sin\theta_{2i}\overrightarrow{w_{4i}}$$

(6)

By considering the expression of the $w_{4i}$ unit vectors in the local frames:

$$\overrightarrow{w_{4i}} = \hat{j}_i$$

$$\overrightarrow{w_{2i}} = \cos\theta_{2i}\hat{i}_i - \sin\theta_{2i}\hat{k}_i$$

(7)

(8)

and taking into account (6), Eq. (5) can be easily expressed in $i^{th}$ leg’s local frame:

$$\begin{pmatrix}
\cos\theta_{i} & \sin\theta_{i} & 0 & p_x & r & s_i & 0 \\
-\sin\theta_{i} & \cos\theta_{i} & 0 & p_y & 0 & 0 & -1 \sin\theta_{2i} \\
0 & 0 & 1 & p_z & 0 & h & 1 \cos\theta_{2i} \cos\theta_{ni} \\
\end{pmatrix}$$

for $i=1,2,3$ (9)

3.1 Inverse Position Analysis

The inverse kinematics problem aims at working out active joints’ displacements, $s_i$, once cartesian coordinates $p_x, p_y, p_z$ have been specified; by solving the non-linear system (9) it is obtained:
\[
\begin{align*}
\theta_i &= \text{atan2}(\text{sign}(\theta_i)(\cos \varphi_i p_x + \sin \varphi_i p_y + r - s_i), \\ & \quad \text{sign}(\theta_i)(p_z - h)) \\
&& \quad \text{for } i=1,2,3 \end{align*}
\]

for \(i=1,2,3\): since for each value of \(\theta_2\) only one \(\theta_1\) has been found, inverse kinematics is characterised by a total number of 4 different configurations for each limb.

Workspace boundaries can be easily detected by letting the radicand in (12) vanish, that leads to:

\[
t^2 - \left(\sin \varphi_i \cdot p_x - \cos \varphi_i \cdot p_y \right)^2 - \left(p_z - h\right)^2 = 0
\]

A picture of the resulting workspace, for the sample geometrical values: \(r=150, l=350, h=0\), is shown in Fig. 3: it has a convex dome-shaped boundary surface that, if cut at varying heights \(p_z\), yields a set of regular hexagons of height \(\sqrt{t^2 - (p_z - h)^2}\).

### 3.2 Direct Position Analysis

In direct kinematics problem the displacements of actuated joints’ variables, \(s_i\), are known and platform’s positions \(p_i\) must be worked out; by squaring the 3 equations in (9) and summing up side by side, it is obtained:

\[
p_i^2 + p_j^2 + p_k^2 + a_i p_x + b_i p_y + c_i p_z + k_i = 0
\]

for \(i=1,2,3\), having defined:

\[
\begin{align*}
a_i &= 2 \cos \varphi_i \cdot (r - s_i) \\
b_i &= 2 \sin \varphi_i \cdot (r - s_i) \\
c_i &= -2 h \\
k_i &= r^2 + h^2 + s_i^2 - 2 r s_i - l^2
\end{align*}
\]

By subtracting Eq. (14) for \(i=1\) from Eq. (12) for \(i=2,3\), respectively, it is obtained:

\[
\begin{align*}
(a_2 - a_1) p_x + (b_2 - b_1) p_y + (c_2 - c_1) p_z + (k_2 - k_1) &= 0 \\
(a_3 - a_1) p_x + (b_3 - b_1) p_y + (c_3 - c_1) p_z + (k_3 - k_1) &= 0
\end{align*}
\]

The algebraic system (16-17) can be solved to find the expressions of \(p_x\) and \(p_y\) as functions of \(p_z\): if they are substituted back in (14) for \(i=1\), a second order polynomial is obtained that provides 2 values for \(p_z\). In correspondence of each solution for \(p_z\), Eqs. (16-17) yield just 1 value for \(p_x\) and \(p_y\), therefore direct kinematics is characterised by two possible different poses.

### 4 Differential kinematics

Deriving Eq. (5) with respect to time, it is obtained:

\[
V_p = \dot{s}_i \mathbf{i}_i + \frac{d}{dt} \left[ B C_i \right]
\]

for \(i=1,2,3\) (18)

(a)

Taking into consideration the expression (6) for \(B C_i\), and performing the needed derivations, Eq. (18) becomes:

\[
V_p = \dot{s}_i \mathbf{i}_i - l \hat{\omega}_j \cos \vartheta_2 \mathbf{w}_u + l \hat{\omega}_j \cos \vartheta_2 \mathbf{w}_j - l \hat{\omega}_2 \sin \vartheta_2 (\mathbf{w}_2 \times \mathbf{w}_u)
\]

(19)

If Eq. (19) is dot-multiplied by \(\mathbf{w}_u \cdot \mathbf{w}_2\) and \((\mathbf{w}_2 \times \mathbf{w}_u)\) orderly, the following three scalar equations are obtained:

\[
\begin{align*}
V_p \cdot \mathbf{w}_u &= -l \hat{\omega}_2 \cos \vartheta_2 \\
V_p \cdot \mathbf{w}_2 &= \dot{s}_i \mathbf{i}_i \cdot \mathbf{w}_2 + l \hat{\omega}_j \cos \vartheta_2 \\
V_p \cdot (\mathbf{w}_2 \times \mathbf{w}_u) &= \dot{s}_i \mathbf{i}_i \cdot (\mathbf{w}_2 \times \mathbf{w}_u) - l \hat{\omega}_2 \sin \vartheta_2
\end{align*}
\]

(20)

(21)

(22)

The set of equations (20-22) can be solved to eliminate the dependence from the derivatives of articular variables, thus arriving at a single scalar equation for each leg in the form:

![Figure 3. Workspace boundary (a) and level curves (b)](image-url)
Equation (23) written three times, once for each \( i=1,2,3 \), yields three scalar equations which can be arranged in the following matrix form:

\[
\begin{bmatrix}
\sin \theta_{1i} \cos \theta_{21} & \sin \theta_{12} \cos \theta_{22} & \sin \theta_{13} \cos \theta_{23} \\
\sin \theta_{1i} \sin \theta_{21} - \cos \theta_{1i} \sin \theta_{21} & \sin \theta_{12} \sin \theta_{22} - \cos \theta_{12} \sin \theta_{22} & \sin \theta_{13} \sin \theta_{23} - \cos \theta_{13} \sin \theta_{23} \\
\cos \theta_{1i} & \cos \theta_{12} & \cos \theta_{13}
\end{bmatrix}
\begin{bmatrix}
\frac{\omega_i}{\sin \theta_{21}} \\
\frac{\omega_j}{\sin \theta_{22}} \\
\frac{\omega_k}{\sin \theta_{23}}
\end{bmatrix}
= \begin{bmatrix}
\frac{\omega_2}{\sin \theta_{21}} \\
\frac{\omega_j}{\sin \theta_{22}} \\
\frac{\omega_k}{\sin \theta_{23}}
\end{bmatrix}
\]

(23)

with:

\[
\frac{\omega_i}{\sin \theta_{21}} = \frac{\omega_j}{\sin \theta_{22}} = \frac{\omega_k}{\sin \theta_{23}}
\]

(24)

where the inverse and direct kinematics Jacobian matrices are given by:

\[
J_i = \begin{bmatrix}
\sin \theta_{1i} \cos \theta_{21} & 0 & 0 \\
0 & \sin \theta_{12} \cos \theta_{22} & 0 \\
0 & 0 & \sin \theta_{13} \cos \theta_{23}
\end{bmatrix}
\]

(25)

\[
J_D = \begin{bmatrix}
J_D^{(1)} & J_D^{(2)} & J_D^{(3)}
\end{bmatrix}
\]

(26)

4.1 Rotation singularities

Since the 3-PUU mechanism has usually 3 dof of pure translation in the 6 dof cartesian space, certain configurations may exist such that the machine becomes liable to rotations of the platform. In order to find out such constraints singularities [12], Eq. (4) is dot-multiplied by \( \mathbf{n}_i = (\mathbf{w}_i \times \mathbf{w}_i) \):

\[
\mathbf{n}_i \cdot \mathbf{\omega} = 0
\]

(27)

Eq. (27), written for \( i=1,2,3 \), represents an algebraic system of 3 scalar equations in the 3 unknowns \( \omega_\theta, \omega_\phi, \omega_\psi \); it only admits the trivial null solution, unless the coefficient matrix \( C \) becomes singular:

\[
C = \begin{bmatrix}
\mathbf{n}_1^T \\
\mathbf{n}_2^T \\
\mathbf{n}_3^T
\end{bmatrix}
\]

(28)

i.e. if its determinant happens to be null:

\[
|C| = \mathbf{n}_1 \cdot (\mathbf{n}_2 \times \mathbf{n}_3)
\]

(29)

By taking into account the expression of the vectors \( \mathbf{w}_i \) and \( \mathbf{w}_j \) in Eqs. (7-8), the matrix \( C \) can be written in function of the articular coordinates \( \theta \):

\[
C = \begin{bmatrix}
\sin \theta_{1i} \cdot \cos \theta_{21} & \sin \theta_{1i} \cdot \sin \theta_{21} & \cos \theta_{21} \\
\sin \theta_{12} \cdot \cos \theta_{22} & \sin \theta_{12} \cdot \sin \theta_{22} & \cos \theta_{22} \\
\sin \theta_{13} \cdot \cos \theta_{23} & \sin \theta_{13} \cdot \sin \theta_{23} & \cos \theta_{23}
\end{bmatrix}
\]

(30)

By substituting the actual value of \( \phi_i \), for \( i=1,2,3 \), an explicit expression for matrix determinant can be obtained:

\[
|C| = \frac{\sqrt{3}}{2} \cdot (\sin \theta_{1i} \sin \theta_{13} \cos \theta_{13} + \sin \theta_{1i} \sin \theta_{12} \cos \theta_{12} + \sin \theta_{1j} \sin \theta_{13} \cos \theta_{13})
\]

(31)

Therefore, rotational singularities may only occur if:

- \( \theta_{1i} = \theta_{1j} = 0 \) with \( i \neq j \);
- \( \theta_{1i} = 90^\circ \) and \( \theta_{1j} = 0 \);
- \( \theta_{1i} = \theta_{12} = \theta_{13} = 90^\circ \).

By considering the expression of \( \theta_{1i} \) coming from position analysis, it is seen that it only vanishes on the upper boundary of the workspace, while it can only be 90° on the lower, flat surface of the workspace (\( \mathbf{p}_z = h \)).

4.2 Translation singularities

When the \( J_i \) matrix becomes singular in (24), certain translational motions cannot be obtained any more and the mechanism is said to be stuck in an inverse kinematics singularity; by posing the determinant of \( J_i \) equal to zero, it is obtained:

\[
|J_i| = \sin \theta_{1i} \cdot \cos \theta_{21} \cdot \sin \theta_{12} \cdot \cos \theta_{22} \cdot \sin \theta_{13} \cdot \cos \theta_{23}
\]

(32)

Eq. (32) leads to the same considerations developed in § 4.1, therefore also inverse kinematics singularities can only appear on the boundary of the workspace.

Turning to direct kinematics singularities, they are characterised by the possibility of platform’s motions also when all the actuators are locked up and can be identified by letting the determinant of \( J_D \) vanish; by substituting in (26) the expressions of \( \theta_{1i} \), worked out in (10-11) and after some tedious computations, the problem is turned into the evaluation of the following determinant:
where the simple coordinate change $z=\left(p_z-h\right)$ has been introduced. By posing to zero the determinant (33) a rather complex function is obtained. Fig. 4 represents the value of the determinant at varying heights $z$; it is observed that the maximum of the function always lies in the centre of the hexagon, then it slowly decreases to reach zero or even negative values in correspondence of workspace boundaries.

Given the regular shapes evidenced in Fig. 4a-d, and in order to have a little more insight in the form of the singularity surfaces, it is checked whether the points of the sphere of equation:

$$z^2 + x^2 + y^2 = l^2$$  \hspace{1cm} (34)

are singular points by direct substitution of (34) into (33); making reference to Fig. 5, it results that in regions 1, 3 and 5 the determinant actually vanishes on the sphere (34), that is internally tangent to the workspace boundary, while in regions 2, 4 and 6 the singularity surfaces are always external to the workspace itself; therefore it can be concluded that the internal space of the sphere of radius $l$ is free of any type of singularities.

The mapping of Eq. (34) in joint space can be easily obtained, so that simple analytical expressions for $s_i, \ i=1,2,3$ can let a controller bring the 3-PUU machine away from singular surfaces during run-time motions.

Figure 4. Determinant of Jacobian matrix $J_D$ at different heights: 25 mm (a), 100 mm (b), 200 mm (c), 325 mm (d)
5 Conclusions

The paper has presented the kinematic analysis of a particular type of 3-PUU mechanism, characterised by a star setting of frame slideways. After having worked out position kinematics, the workspace of the mechanism has been identified: it is convex and nearly hemispheric. Then differential kinematics has been performed, with the investigation of singular configurations: it resulted that they all lie close to workspace boundary, therefore the occurrence of singularities can be easily prevented by properly setting the lengths of frame slideways.

The variant of the 3-PUU mechanism that has been here discussed has a different setting with respect to all other configurations that have been studied in literature so far: also the resulting kinematics properties are quite different from other machines of this kind, let alone the properties of different architectures like the 3-UPU or the Linear Delta. Therefore it is Authors’ opinion that some applications could benefit of this kind of architecture as a base for the synthesis of automated machines.

References


